Therefore

5
$$\boxed{\lim_{x \to 0} (1 + x)^{1/x} = e}$$

Formula 5 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

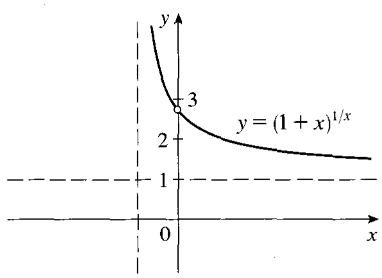

FIGURE 4

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

6
$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

3.8 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–20 □ Differentiate the function.

2. $f(x) = \ln(2 - x)$
3. $f(\theta) = \ln(\cos\theta)$
4. $f(x) = \cos(\ln x)$
5. $f(x) = \log_3(x^2 - 4)$
6. $f(x) = \log_{10}\left(\dfrac{x}{x - 1}\right)$
7. $F(x) = \ln\sqrt{x}$
8. $G(x) = \sqrt[3]{\ln x}$
9. $f(x) = \sqrt{x}\ln x$
10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$
11. $g(x) = \ln\dfrac{a - x}{a + x}$
12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$
13. $F(x) = e^x \ln x$
14. $h(y) = \ln(y^3 \sin y)$
15. $y = \dfrac{\ln x}{1 + x}$
16. $y = (\ln\tan x)^2$
17. $y = \ln|x^3 - x^2|$
18. $G(u) = \ln\sqrt{\dfrac{3u + 2}{3u - 2}}$
19. $y = \ln(e^{-x} + xe^{-x})$
20. $y = \ln(x + \ln x)$

21–24 □ Find y' and y''.

21. $y = x\ln x$
22. $y = \ln(1 + x^2)$
23. $y = \log_{10} x$
24. $y = \ln(\sec x + \tan x)$

25–28 □ Differentiate f and find the domain of f.

25. $f(x) = \ln(2x + 1)$

26. $f(x) = \dfrac{1}{1 + \ln x}$

27. $f(x) = x^2 \ln(1 - x^2)$

28. $f(x) = \ln \ln \ln x$

29. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

30. If $f(x) = x^2 \ln x$, find $f'(1)$.

31–32 □ Find an equation of the tangent line to the curve at the given point.

31. $y = \ln \ln x$, $(e, 0)$

32. $y = \ln(x^2 + 1)$, $(1, \ln 2)$

33. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

34. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

35–46 □ Use logarithmic differentiation to find the derivative of the function.

35. $y = (2x + 1)^5(x^4 - 3)^6$

36. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$

37. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

38. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

39. $y = x^x$

40. $y = x^{1/x}$

41. $y = x^{\sin x}$

42. $y = (\sin x)^x$

43. $y = (\ln x)^x$

44. $y = x^{\ln x}$

45. $y = x^{e^x}$

46. $y = (\ln x)^{\cos x}$

47. Find y' if $y = \ln(x^2 + y^2)$.

48. Find y' if $x^y = y^x$.

49. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

50. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

51. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

52. Show that $\displaystyle\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

3.9 Hyperbolic Functions

Certain combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and cosine can be sketched using graphical addition as in Figures 1 and 2.